Simultaneous Linear Differential Equations

SIMULTANEOUS DIFFERENTIAL EQUATIONS

If two or more dependent variables are functions of a single independent variable, the equations involving their derivatives are called simultaneous equations, e.g.,

$$\frac{dx}{dt}+4y=t$$

$$\frac{dy}{dt}+2x=e^t$$

The method of solving these equations is based on the process of elimination, as we solve algebraic simultaneous equations.

Q.1 The equations of motions of a particle are given by

$$\frac{dx}{dt}+w\,y=0$$

$$\frac{dx}{dt}-w\,x=0$$

Find the path of the particle and show that it is a circle.

Sol. On putting $\frac{d}{dt}\equiv D$ in the equations, we have

$$Dx+wy=0 \qquad \ldots(1)$$

$$-wx+Dy=0 \qquad \ldots(2)$$

On multiplying (1) by w and (2) by D, we get

$$wDx+w^2y=0 \qquad \ldots(3)$$

$$-wDx+D^2y=0 \qquad \ldots(4)$$

On adding (3) and (4), we obtain

$$w^2 y + D^2 y = 0 \qquad \Rightarrow \qquad (D^2 + w^2) y = 0 \qquad \ldots(5)$$

Now, we have to solve (5) to get the value of y.

A.E. is $D^2 + w^2 = 0 \quad \Rightarrow \quad D^2 = -w^2 \quad \Rightarrow \quad D = \pm iw$

$$y = A\cos wt + B\sin wt \qquad \ldots(6)$$

$$\Rightarrow \qquad Dy = -A\omega \sin \omega t + B\omega \cos wt$$

On putting the value of Dy in (2), we get

$$-wx - Aw\sin wt + Bw\cos wt = 0$$

$$\Rightarrow \qquad wx = -Aw\sin wt + Bw\cos wt$$

$$\Rightarrow \qquad x = -A\sin wt + B\cos wt$$

On squaring (6) and (7) and adding, we get

$$x^2 + y^2 = A^2(\cos^2 wt + \sin^2 wt) + B^2(\cos^2 wt + \sin^2 wt)$$

$$\Rightarrow \qquad x^2 + y^2 = A^2 + B^2$$

This is the equation of circle.

Q.2 Solve the following

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dx}{dt} = -x + -3y$$

With $x(0) = 6$ and $y(0) = -2$.

Sol. Here we have

$$\frac{dx}{dt} = 3x + 8y \qquad \ldots(1)$$

$$\frac{dx}{dt} = -x - 3y \qquad \ldots(2)$$

On putting $\frac{d}{dt} \equiv D$ in (1) and (2), we get

$$Dx - 3x - 8y = 0 \quad \Rightarrow \quad (D-3)x - 8y = 0 \qquad \ldots(3)$$

and $$Dy + x + 3y = 0 \quad \Rightarrow \quad (D+3)y + x = 0 \qquad \ldots(4)$$

Multiplying (3) by $(D+3)$ and (4) by 8 adding them we get

$$(D^2 - 1)x = 0$$

A.E. is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m^2 - 1 \quad \Rightarrow m \ \pm\ 1$

$\therefore \quad C.F. = C_1 e^t + C_2 e^{-t}$

$P.I. = 0$

$$\therefore \quad x = C_1 e^t + C_2 e^{-t} \qquad \ldots(5)$$

From (3) we get $(D-3)[C_1 e^x + C_2 e^{-x}] = 8y$

$$\Rightarrow 8y = -C_1 e^t - C_2 e^{-t} - 3C_1 e^t - 3C_2 e^{-t}$$

$$\Rightarrow 8y = -2C_1 e^t - 4C_2 e^{-t}$$

$$\Rightarrow y = -\frac{1}{2}(C_1 e^t + 2C_2 e^{-t}) \qquad \ldots(6)$$

Initially when $t = 0$ then $x = 2.$

From (5); $2 = C_1 e^0 + C_2 e^0 \Rightarrow C_1 + C_2 = 2$

Also when $t = 0$, $y = -2$.

From (6), $$-2 = -\frac{1}{4}(C_1 e^0 + 2C_2 e^0)$$

$$\Rightarrow 8 = C_1 + 2C_2$$

Solving (7) and (8), we get

$$C_1 = -4 \text{ and } C_2 = 6$$

Hence, the required solution is

$$x = -4e^t + 6e^t$$

and $y = -\frac{1}{4}(-4e^t + 12e^t)$

Q.3 Solve : $\frac{d^2x}{dt^2} + y = \sin t,$ $\frac{d^2y}{dt^2} + x = \cos t,$

Sol. Here, we have

$$D^2x + y = \sin t \qquad \ldots(1)$$

$$x + D^2y = \cos t \qquad \ldots(2) \quad \left(D \equiv \frac{d}{dt}\right)$$

Operating equation (1) by D^2, we get

$$D^4x + D^2y = -\sin t \qquad \ldots(3)$$

Subtracting (2) from (3), we get

$$(D^4 - 1)x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0 \quad \Rightarrow \quad (m^2 - 1)(x^2 + 1) = 0$$

$$\Rightarrow \quad m = 1, -1, \pm i$$

$$C.F. = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$$

$$P.I. = \frac{1}{D^4 - 1}(\sin t - \cos t)$$

$$= -t.\frac{1}{4D^3}(\sin t + \cos t)$$

$$= \frac{t}{4}(-\cos t + \sin t) \qquad \left(\frac{1}{D} \equiv \int\right)$$

$$\therefore \quad x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{1}{4}(\sin t - \cos t) \qquad \ldots(4)$$

From (1), $y = \sin t - \dfrac{d^2 x}{dt^2}$

$$= \sin t - \frac{d^2}{dt^2}\left[(c_1 e^t + c_2 e^{-t}) + (c_3 \cos t + c_4 \sin t) + \frac{1}{4}(\sin t - \cos t)\right]$$

$$= \sin t - \frac{d}{dt}\left[c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right]$$

$$= \sin t - \left[c_1 e^t - c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{1}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)\right]$$

$$y = -c_1 e^t - c_2 e^{-t} - c_3 \cos t + c_4 \sin t + \frac{1}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t)$$

and $x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \dfrac{1}{4}(\sin t - \cos t)$ [From (4)] Ans.